Last time...

Goal: Construct Θ-stratifications
$$\mathcal{X} = \mathcal{X}^{ss} \cup \bigcup_\alpha S_\alpha$$
along with good moduli space $q: \mathcal{X}^{ss} \to M$

Keywords from first lecture:
filtrations $f: \Theta \to \mathcal{X}$, numerical invariant $\mu(f)$, stability, HN problem

Thm A (HL): if $\mathcal{X}$ Θ-reductive, HN problem has a solution, and only finitely many HN types in bounded family $\Longrightarrow$ μ defines Θ-stratification

Thm B (Alper-HL-Heinloth): if $\mathcal{X}$ is bounded, it has a GMS iff
1) $\mathcal{X}$ is Θ-reductive, 2) unpunctured inertia,
3) closed points have reductive autom. groups.

Today: Discuss this theorem and applications to moduli of sheaves on a K3 surface

What is a Θ-reductive stack?

Def: A stack is Θ-reductive if...
For any family over DVR $\mathrm{Spec}(R) \to \mathcal{X}$, any filtration of the generic point extends to a filtration of the family

Ex 1: Y projective scheme, $\mathcal{X} = \mathrm{Coh}(Y)$
$\rightsquigarrow$ amounts to compactness of flag scheme
$\rightsquigarrow$ fails for $\mathrm{Bun}(Y)$

Ex 2: more generally, proof can be adapted to $\mathcal{X} = \{\text{objects in } \mathcal{A}\}$, $\mathcal{A}$ abelian category

Ex 3: quotient stacks $\mathrm{Spec}(A)/G$, G reductive

Prop: if $\mathcal{X}$ is Θ-reductive and a numerical invariant μ defines a Θ-stratification, then $\mathcal{X}^{ss}$ is Θ-reductive.

Modifications and unpunctured inertia

> Def: given a family over a DVR $\operatorname{Spec}(R) \to \mathfrak{X}$, we say that another map is a modification if the maps are isomorphic over the generic point $\operatorname{Spec}(K)$

Ex: family of bundles on $C \times \operatorname{Spec}(R)$, $\mathcal{E}_0 =$ special fiber, $\mathcal{F} \subset \mathcal{E}_0$ sub-bundle

$$\mathcal{E}' = \ker(\mathcal{E} \to i_*(\mathcal{E}_0/\mathcal{F}))$$

new bundle, elementary modification

$\rightsquigarrow$ can formulate a notion for arbitrary $\mathfrak{X}$

Def: $\mathfrak{X}$ has unpunctured inertia if for any family over a DVR, one can find an elementary modification such that

1) any connected component of $\operatorname{Aut}(\text{generic fiber})$ specializes to special fiber, or

2) any finite-order generic automorph. specializes

Amplifications

If $q: \mathfrak{X} \to M$ is a good moduli space, then

$\longrightarrow$ M is separated iff any modification over a DVR can be factored into sequence of elementary modifications

$\longrightarrow$ M is proper if it is separated and $\mathfrak{X}$ satisfies existence part of val. crit.

Thm (semistable reduction): Given a Θ-stratification of $\mathfrak{X}$, any family $\operatorname{Spec}(R) \to \mathfrak{X}$ with semistable generic point is related by elementary modification to a semistable family

Consequence: Can specify conditions on $\mathfrak{X}$ s.t. the good moduli space of $\mathfrak{X}^{ss}$ is proper

Slope semistability

Set up: $\sigma \left[\begin{array}{l} \mathcal{A} \subset D^b(Y) \text{ heart of bounded } t\text{-structure} \\ v : K_0(Y) \longrightarrow \Lambda = K_0^{num}(Y) \\ Z : \Lambda \longrightarrow \mathbb{C}, \quad -\deg(v) + i\,\mathrm{rk}(v) := Z(v) \end{array}\right.$

Pre-stability condition $\rightsquigarrow$ all $E \in \mathcal{A}$ have HN filtrations

Hypothesis:
$$\mathcal{X}_v(B) := \left\{ \begin{array}{l} E \in D^b(Y \times B) \text{ s.t.} \\ E|_b \in \mathcal{A} \text{ for all } b \in B \end{array} \right\}$$
is an algebraic stack

Ex 1: Slope semistability in $\mathrm{Coh}(Y)$

Ex 2: Any Bridgeland stability condition with $\mathcal{A}$ noetherian

Rem: can work in more general categories, and can define $\mathcal{X}$ directly from $\mathcal{A}$

Moduli spaces

Central charge defines a line bundle: on $\mathcal{X}_v$

Write $Z(E) = \chi(E \otimes \omega_Z)$, $\omega_Z \in K_0^{num}(Y) \otimes \mathbb{C}$

$$\begin{tikzcd} & \mathcal{X} \times Y \arrow[dl] \arrow[dr] & \\ \mathcal{X} & & Y \end{tikzcd}$$

$$\ell \propto ch_1\left(\Phi^{Y \to \mathcal{X}}_{\mathcal{E}_{univ}} \left(\mathrm{Im}\left(\frac{-\omega_Z}{Z(v)} \right) \right) \right)$$
$$b \propto ch_2\left(\Phi^{Y \to \mathcal{X}}_{\mathcal{E}_{univ}} \left(\mathrm{Im}(\omega_Z) \right) \right)$$

Consequences of main theorems: if $\mathcal{X}_v^{ss}$ bounded $\forall v \in \Lambda$, then

$\longrightarrow \mathcal{X}$ has Θ-stratification by HN type

$\longrightarrow \mathcal{X}_v^{ss}$ has proper good moduli space

Main example: From now on, we consider only Bridgeland stability on a smooth surface

$\rightsquigarrow$ can study Donaldson invariants

Naïve Donaldson invariants of surfaces

We will always work with K-theoretic invariants:

→ $\mathcal{F} \in K^0(\mathfrak{X})$, e.g. $\mathcal{F} = \Phi_{\mathcal{E}}^{Y\to\mathfrak{X}}(E)$
where $E \in K^0(Y)$, $\mathcal{E}$ obtained from $\mathcal{E}_{univ}$ by simple operations

→ $Rq_* : D^b(\mathfrak{X}_v^{ss}) \to D^b(M_v^{ss})$ is well-defined by properties of GMS

Definition

$$\boxed{I_v^\sigma(\mathcal{F}) := \chi(M_v^{\sigma\text{-}ss}, Rq_*(\mathcal{F})) = \chi(\mathfrak{X}_v^{\sigma\text{-}ss}, \mathcal{F})}$$

Question: how do $\mathfrak{X}_v^{\sigma\text{-}ss}$ and $I_v^\sigma(E)$ depend on stability condition σ? For nice results, we need to regard $\mathfrak{X}_v^{\sigma\text{-}ss}$ as a derived stack.

→ Algebraic geometry built commutative DGA's
e.g. $A[\underbrace{\epsilon_1, \dots, \epsilon_r}_{\deg 1} ; d\epsilon_i = a_i \in A]$

→ $I_v^\sigma(\mathcal{F})$ = integral over derived stack

Correct Donaldson invariants of surfaces

A simple analogy for derived algebraic geometry:

reduced rings ⇝ rings ⇝ CDGA's

On affine objects:

assume homologically bounded

$$H_0(A_\bullet)^{red} \longleftarrow H_0(A_\bullet) \longleftarrow A_\bullet$$

Analogous picture for derived schemes / stacks: $i: \mathfrak{X}^{cl} \hookrightarrow \mathfrak{X}$, same underlying points

Virtual structure sheaf: Note $\bigoplus H_i(A_\bullet)$ is a coherent $H_0(A_\bullet)$-module
⇝ glue to define $\mathcal{O}_{\mathfrak{X}}^{vir} := \bigoplus H_i(A_\bullet)[i] \in D^b(\mathfrak{X}^{cl})$

Classical shadow of derived world:

Given $\mathcal{F}$ on derived stack $\mathfrak{X}$,

$$\chi(\mathfrak{X}, \mathcal{F}) = \chi(\mathfrak{X}, \mathcal{F} \otimes i_*(\mathcal{O}_{\mathfrak{X}}^{vir}))$$
$$= \chi(\mathfrak{X}^{cl}, Li^*(\mathcal{F}) \otimes^L \mathcal{O}_{\mathfrak{X}}^{vir})$$

Wall crossing

Situation: Fix $v \in K_0^{num}(S)$

→ σ varies in a complex manifold $\mathrm{Stab}^*(S)$

→ in complement of real codim 1 walls, $\mathcal{X}_v^{\sigma\text{-ss}}$ is constant

→ let $\sigma_0 \in$ wall, $\sigma_\pm$ in different chambers, $\mathcal{X}_v^{\sigma_0\text{-ss}}$ has GMS, and $\mathcal{X}_v^{\sigma_+\text{-ss}} \dashrightarrow \mathcal{X}_v^{\sigma_-\text{-ss}}$ in some cases, as in last lecture

$$\boxed{\mathcal{X}_v^{\sigma_0\text{-ss}} = \mathcal{X}_v^{\sigma_\pm\text{-ss}} \cup \bigcup_\alpha S_\alpha^\pm \quad \alpha \in \left\{ \begin{matrix} v_1 + \cdots + v_p = v \\ Z_{\sigma_0}(v_i) \in \mathbb{R}_{>0} Z_{\sigma_0}(v) \end{matrix} \right\}}$$

Hypothesis: $\forall E \in \mathcal{A}$, $L^{vir}_{\mathcal{X},E} = \mathrm{RHom}(E, E[1])^*$ has no cohomology in deg <-1, i.e. $\mathrm{Hom}(E, E[i]) = 0$ for $i > 2$

Rem: Holds automatically when $\mathcal{A} = \mathrm{Coh}(S)$ or when S is K3

Wall crossing formula

Thm: Under the previous hypotheses, we have

$$I_v^{\sigma_0}(\mathcal{F}) - I_v^{\sigma_\pm}(\mathcal{F}) = \sum_\alpha \chi\left(\mathcal{X}_{v_1}^{\sigma_\pm\text{-ss}} \times \cdots \times \mathcal{X}_{v_p}^{\sigma_\pm\text{-ss}}, \mathcal{F} \otimes E_\alpha^\pm\right)$$

↑ centers of strata

Decompose $L^{vir}_{\mathcal{X}}|_{center} \cong L^+ \oplus L^0 \oplus L^-$

$$E_\alpha = \mathrm{Sym}(L^-) \otimes \mathrm{Sym}((L^+)^*) \otimes \det((L^+)^*)[-\mathrm{rk}(L^+)]$$

Proof idea: Local cohomology, uses derived AG, and modular interpretation of the strata

Uses: 1) compute $I_v^{\sigma_+}(\mathcal{F}) - I_v^{\sigma_-}(\mathcal{F})$

2) come up with explicit formulas for $I_v^{\sigma_+}$ by wall crossing to where $\mathcal{X}_v^{ss} = \emptyset$

Nagging question: Combinatorial structure?

Birational geometry -- K3 case

If v is primitive and σ generic, then

$$M_v^{\sigma\text{-ss}} = \text{smooth projective hyperkähler}$$

Restrict to class of CY manifolds birationally equivalent to $M_v^{\sigma-\mathrm{ss}}$

> Thm (Bayer-Macri): Any two manifolds in this class can be connected by a sequence of birational modifications of the form:
>
> $$M_v^{\sigma_+\text{-ss}} \dashrightarrow M_v^{\sigma_-\text{-ss}}$$
>
> For some (twisted) K3 surface S

We now have a diagram:

$$\begin{tikzcd} M_v^{\sigma_+\text{-ss}} \arrow[rr, dashed] \arrow[dr] & & M_v^{\sigma_-\text{-ss}} \arrow[dl] \\ & M_v^{\sigma_0\text{-ss}} & \end{tikzcd}$$

Local models for flops

Base change the picture:

$$\begin{tikzcd} \mathfrak{X}_v^{\sigma_0\text{-ss}} \arrow[d] & \mathrm{Spec}(A)/G \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\ M_v^{\sigma_0\text{-ss}} & \mathrm{Spec}(A^G) \arrow[l, "\text{étale}"'] \end{tikzcd}$$

One can show, using self-duality $L_{\mathfrak{X}} \cong L_{\mathfrak{X}}^*$

> Thm: Spec(A) is zero fiber of a "weak" algebraic moment map $\mu: \mathrm{Spec}(B) \to \mathfrak{g}^\vee$ for a smooth affine G-scheme.

Application: we recently used this to prove that any two smooth projective CY manifolds in birational class of $M_v^{\sigma-\mathrm{ss}}$ have equivalent derived categories of coherent sheaves.